Student Number: ______________________

Teacher: ______________________

Class: ______________________

FORT STREET HIGH SCHOOL

2018

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 1

Time allowed: 2 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10
HE2, HE4	Manipulates algebraic expressions to solve problems from topic areas such as inverse functions, trigonometry, polynomials, permutations and combinations.	11, 12
HE3, HE5 HE6	Uses a variety of methods from calculus to investigate mathematical models of real life situations, such as projectiles, kinematics and growth and decay	13
HE7	Synthesises mathematical solutions to harder problems and communicates them in appropriate form	14

Total Marks 70

Section I 10 marks
Multiple Choice, attempt all questions,
Allow about 15 minutes for this section

Section II 60 Marks
Attempt Questions 11-14,
Allow about 1 hour 45 minutes for this section

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 60	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
	Percent	

General Instructions:

- Questions 11-14 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

SECTION I (One mark each)

Answer questions 1 to 10 on the multiple choice answer sheet.

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1 The number plates for motor vehicles consist of 2 letters, 2 numbers and then 2 letters such as AB01CD. How many different number plates are possible?

(A) 135 200

(B) 270 400

(C) 37 015 056

(D) 45 697 600

2 What is the exact value of $\tan 75^\circ$?

(A) $2-\sqrt{3}$

(B) $4-\sqrt{3}$

(C) $2+\sqrt{3}$

(D) $4+\sqrt{3}$

3 A curve has parametric equations $x=\frac{2}{t}$ and $y=2t^2$.

What is Cartesian equation of this curve?

(A) $y=\frac{4}{x}$

(B) $y=\frac{8}{x}$

(C) $y=\frac{4}{x^2}$

(D) $y=\frac{8}{x^2}$

4 What are the coordinates of the point P that divides internally the interval joining the points $A(1,2)$ and $B(7,5)$ in the ratio 2:1?

(A) $(3,3)$

(B) $(3,4)$

(C) $(5,3)$

(D) $(5,4)$

5 What is the solution to the inequality $\dfrac{3}{x-2} \le 4$?

(A) $x < -2$ and $x \ge -\dfrac{11}{4}$

(B) $x > -2$ and $x \le -\dfrac{11}{4}$

(C) $x < 2$ and $x \ge \dfrac{11}{4}$

(D) $x > 2$ and $x \le \dfrac{11}{4}$

6 What is the acute angle to the nearest degree between the lines $y = 1 - 3x$ and $4x - 6y - 5 = 0$?

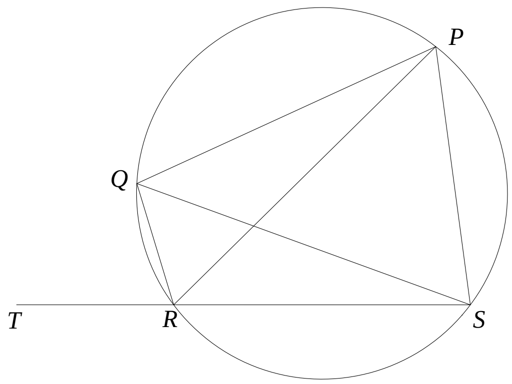
(A) 15°

(B) 38°

(C) 52°

(D) 75°

7 PQRS is a cyclic quadrilateral. SR is produced to T and $\angle PRS = \angle QRT$.



Why is $\angle PQS = \angle PRS$?

(A) Angle at the circumference is equal to the angle in the alternate segment.

(B) Angle between the tangent and a chord is equal to the angle in the alternate segment.

(C) Angle between the two chords in the same segment are equal.

(D) Angles in the same segment standing on the same arc are equal.

8 Consider the polynomial $P(x) = 3x^3 + 3x + a$.
If $x - 2$ is a factor of $P(x)$, what is the value of a?

(A) -30

(B) -18

(C) 18

(D) 30

9 Which of the following is an expression for $\int \cos^2 2x\ dx$?

(A) $x - \frac{1}{4}\sin 4x + c$

(B) $x + \frac{1}{4}\sin 4x + c$

(C) $\frac{x}{2} - \frac{1}{8}\sin 4x + c$

(D) $\frac{x}{2} + \frac{1}{8}\sin 4x + c$

10 A particle moving in a straight line obeys $v^2 = -x^2 + 2x + 8$ where x is its displacement from the origin in metres and v is its velocity in m/s. The motion is simple harmonic. What is the amplitude?

(A) 2π metres

(B) 3 metres

(C) 8 metres

(D) 9 metres

SECTION II (15 marks for each question.)

Answer each question in the appropriate booklet. Extra writing booklets are available.

Question 11: *Use a separate writing booklet .*

a) The letters of the word MOUSE are to be rearranged.

i. How many arrangements are there which start with the letter M and end with the letter E? **1**

ii. How many arrangements are there in which the vowels are grouped together? (A vowel is one of the letters A, E, I, O, U) **1**

iii. How would your answers to parts (i) and (ii) change if the given word had been MOOSE instead of MOUSE? **1**

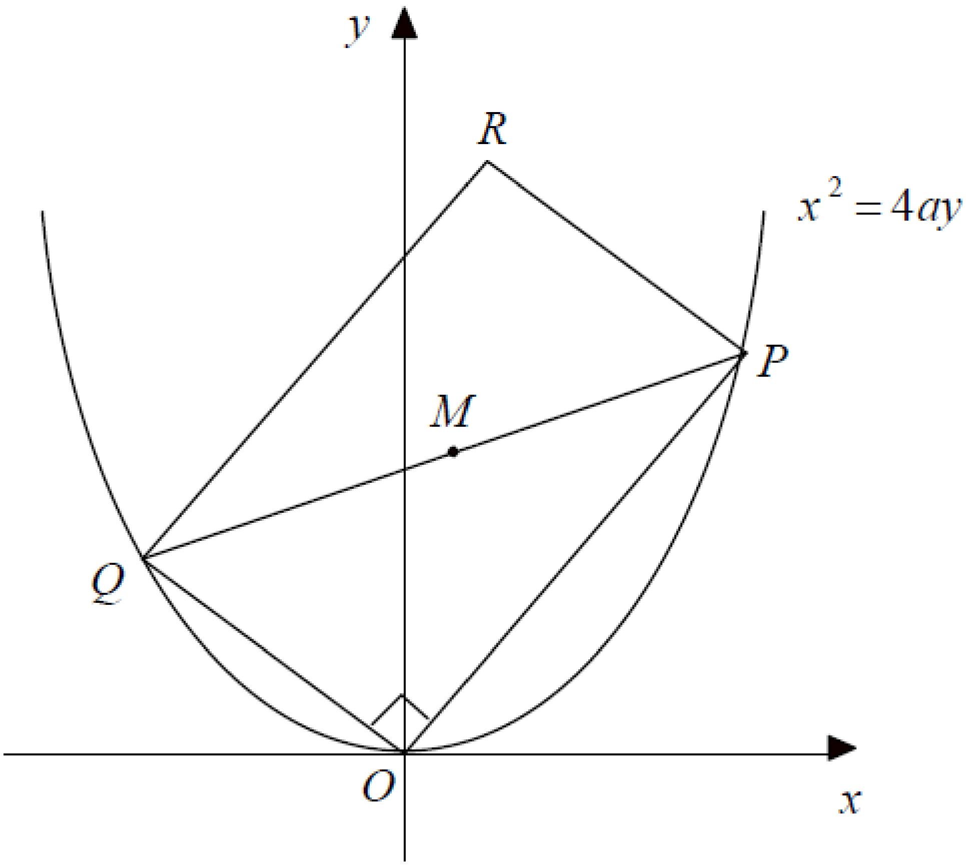
b) Write $7.\dot{1}\dot{2}$ as the sum of an infinite series. Hence write $7.\dot{1}\dot{2}$ as a mixed fraction. **2**

c) Given the function $g(x)=\sqrt{x+2}$ and that $g^{-1}(x)$ is the inverse function of $g(x)$, find $g^{-1}(5)$. **2**

d) Solve the equation $1+\cos 2x=\sin 2x$ for $0\le x\le 2\pi$. **4**

Question 11 continues over the page.

e)



$P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are two points which move on the parabola $x^2 = 4ay$ such that $\angle POQ = 90°$, where $O(0,0)$ is the origin. $M\left(a(p+q), \frac{a}{2}\left(p^2+q^2\right)\right)$ is the midpoint of PQ. R is the point such that $OPRQ$ is a rectangle.

i. Show that $pq = -4$. **1**

ii. Show that R has coordinates of $\left(2a(p+q), a\left(p^2+q^2\right)\right)$. **1**

iii. Find the equation of the locus of R. **2**

Question 12: *Use a separate writing booklet.*

a) The polynomial equation $P(x) = 3x^3 - 2x^2 + x - 4$ has roots α, β and γ. Find the exact value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$. **2**

b) Find $\lim_{h \to 0} \left(\frac{\cos 2h - 1}{h} \right)$. **2**

c) .

i. Sketch the graph of $y = |2 - x|$. **1**

ii. Using this graph, or otherwise, find the solution to $|2 - x| < x$. **2**

d) Use the method of Mathematical induction to show that

$$2 + 5 + 8 + ... + (3n - 1) = \frac{n(3n+1)}{2}$$ **3**

e) A plate is initially heated to $55°C$, and it then cools to $41°C$ in 10 minutes. Assume Newtons Law of Cooling $\frac{dT}{dt} = -k(T - S)$ applies, where S is the surrounding temperature. If the surrounding temperature is $22°C$:

i. Show that the temperature T is given by $T = 22 + 33e^{\frac{-t}{10}\ln\left(\frac{33}{19}\right)}$. **2**

ii. Find the temperature after 25 minutes (to the nearest degree). **1**

iii. Find the time for the plate to cool to $25°C$ (to 1 decimal place). **2**

Question 13: *Use a separate writing booklet.*

a) Find $\frac{d}{dx}\left(e^{-x}\cos^{-1}x\right)$. **2**

b) By making the substitution $u = x - 2$, evaluate $\int_4^5 \frac{x(x-4)}{(x-2)}dx$. **3**

c) For the polynomial $P(x) = x^3 + 2x^2 - 15x - 36$,

i. Factorize $P(x)$ fully over the real numbers. **3**

ii. Hence solve $x^3 + 2x^2 - 15x - 36 < 0$. **2**

d) Wheat falls from an auger onto a conical pile at the rate of 20 cm^3/s. The radius of the base of the pile is always equals to half its height.

i. Show that $V = \frac{\pi}{12}h^3$ and hence find $\frac{dh}{dt}$. **2**

ii. Find the rate, in terms of π, at which the pile is rising when it is 8cm high. **1**

iii. Find the time, to 1 decimal place, for the pile to reach a height of 8cm. **2**

Question 14: *Use a separate writing booklet.*

a) A particle moves in a straight line on the x axis. At time t its velocity is v and its acceleration is $\ddot{x}$. Given $\ddot{x} = 4 - 4x$ and initially $x = 3$ when $v^2 = 20$:

i. Show that the motion is SHM and state the centre of motion and the value of n. **2**

ii. Hence show that $v^2 = 32 + 8x - 4x^2$. **1**

iii. Use this expression for v^2 to find the possible values for x . **1**

iv. Describe the motion of the particle until it first stops if initially $v = -2\sqrt{5}$. **1**

b)

i. Differentiate $x \sin^{-1} x$, and hence find an equation to evaluate $\int \sin^{-1} x \, dx$. **2**

ii. Hence evaluate $\int_0^1 \sin^{-1} x \; dx$. **2**

c) An object is projected from the origin O with initial speed U m/s at an angle of elevation of α . At the same instant another object is projected from a point A which is h units above the origin O. The second object is projected with initial speed V m/s at an angle of elevation of β , where $\beta < \alpha$. Both objects move freely under gravity in the same plane.

i. Given that the equations of motion for the object projected from the origin are:

$$x = Ut \cos \alpha \quad y = Ut \sin \alpha - \frac{1}{2} gt^2 ,$$

write down the equations of motion for the object projected from the point A. **2**

ii. If the objects collide T seconds after they are simultaneously projected, show that $T = \dfrac{h \cos \beta}{U \sin(\alpha - \beta)}$ **4**

End of examination.

Multiple Choice:

1	$\text{Number of arrangements} = 26^2 \times 10^2 \times 26^2$ $= 45\ 697\ 600$	1 Mark: D
2	$\tan 75^\circ = \tan(45^\circ + 30^\circ)$ $= \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$ $= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ $= \dfrac{\sqrt{3}+1}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$ $= 2 + \sqrt{3}$	1 Mark: C
3	$x = \dfrac{2}{t}$ or $t = \dfrac{2}{x}$ Substitute $\dfrac{2}{x}$ for x into $y = 2t^2$ $y = 2(\dfrac{2}{x})^2$ $= \dfrac{8}{x^2}$	1 Mark: D
4	$x = \dfrac{mx_2 + nx_1}{m+n}$ $\quad y = \dfrac{my_2 + ny_1}{m+n}$ $= \dfrac{2 \times 7 + 1 \times 1}{2+1} = 5$ $\quad = \dfrac{2 \times 5 + 1 \times 2}{2+1} = 4$ The coordinates of P are $(5, 4)$	1 Mark: D
5	$(x-2)^2 \times \dfrac{3}{(x-2)} \le 4 \times (x-2)^2$ $(x-2)3 \le 4(x-2)^2 \quad x \ne 2$ $(x-2)(3-4x+8) \le 0$ $(x-2)(11-4x) \le 0$ $x < 2$ and $x \ge \dfrac{11}{4}$	1 Mark: C

6	For $y=1-3x$ then $m_1=-3$ For $4x-6y-5=0$ then $m_2=\frac{2}{3}$ $\tan\theta=\left\|\frac{m_1-m_2}{1+m_1m_2}\right\|$ $=\left\|\frac{-3-\frac{2}{3}}{1+-3\times\frac{2}{3}}\right\|$ $=\left\|\frac{-\frac{11}{3}}{-1}\right\|$ $=\frac{11}{3}$ $\theta=74.7448813..\approx 75^\circ$	1 Mark: D
7	$\angle PQS=\angle PRS$ (angles in the same segment standing on the same arc are equal).	1 Mark: D
8	$P(x)=3x^3+3x+a$ $P(2)=3(2)^3+3(2)+a=0$ ($x-2$ is a factor of $P(x)$) $24+6+a=0$ $a=-30$	1 Mark: A
9	$\int\cos^2 2x\,dx=\int\frac{1}{2}(1+\cos 4x)dx$ $=\frac{1}{2}\left[x+\frac{1}{4}\sin 4x\right]+c$ $=\frac{x}{2}+\frac{1}{8}\sin 4x+c$	1 Mark: D
10	$v^2=-x^2+2x+8$ $=1^2(8+2x-x^2)$ $=1^2(9-(x-1)^2)$ $=n^2(a^2-x^2)$ $a^2=9$ $a=3$ Amplitude is 3 metres	1 Mark: B

Question 11:

a)

i. $1\times 3!\times 1$ ❶

$=6$

ii. Grouping vowels is one item, leaves 3! ways to the vowel group and the 2 consonants. But there are 3! arrangements of the vowels, hence $3!\times 3!=36$ ways. ❶

iii. Both answers would have OU and UO count as one, hence both answers would be divided by 2. ❶

Marking: Reasoning must be clear

Comments: Question asked *how* the answer changed, not what the new answer was.

b) $7.\dot{1}\dot{2}=7+0.12+0.0012+0.000012+...$ which is a GP with $a=0.12, r=0.01$ ❶. Thus

$$7.\dot{1}\dot{2}=7+\frac{a}{1-r}$$
$$=7+\frac{0.12}{1-0.01}$$
$$=7\frac{4}{33}$$ ❶

Marking: Must show a GP as reqd by question or no marks

Comments: Many students did not follow the instructions and hence gained no marks.

c) $g^{-1}(x)$:

$$x=\sqrt{y+2}$$
$$x^2=y+2$$
$$y=x^2-2$$
$$g^{-1}(x)=x^2-2$$ ❶
$$g^{-1}(5)=5^2-2$$
$$=23$$ ❶

d)
$$1+\cos 2x=\sin 2x$$
$$2\cos^2 x=2\sin x\cos x$$ ❶
$$\cos^2 x-\sin x\cos x=0$$
$$\cos x(\cos x-\sin x)=0$$ ❶

Hence:

$\cos x=0$

$x=\frac{\pi}{2},\frac{3\pi}{2}$ ❶

$\cos x-\sin x=0$

$\cos x=\sin x$

$1=\tan x$

$x=\frac{\pi}{4},\frac{5\pi}{4}$ ❶

$$\therefore x=\frac{\pi}{4},\frac{\pi}{2},\frac{5\pi}{4},\frac{3\pi}{2}\left(0\le x\le 2\pi\right)$$

Comments: Dividing by $2\cos x$ was a common error – this lost one set of solutions!

As many students attempted the more complex $R\sin(2x-\theta)$ or $R\cos(2x+\theta)$ transformations, they are duplicated at the end of the solutions.

e)

i. Gradient OP: $m_{OP}=\frac{ap^2-0}{2ap-0}=\frac{p}{2}$ and gradient OQ: $m_{OP}=\frac{aq^2-0}{2aq-0}=\frac{q}{2}$

Comments: Many found p and q for gradients (the tangents) in error.

Solution	Marking	Comments
As $OP \perp OQ$, $m_{OP} \times m_{OQ} = -1$ $\frac{p}{2} \times \frac{q}{2} = -1$ ❶ $pq = -4$		
ii. The diagonals of the rectangle bisect each other, hence *M* is also the midpoint of *OR*. Thus $x_M = \frac{x_R + 0}{2}$ $\quad y_M = \frac{y_R + 0}{2}$ $x_R = 2x_M$ $\quad y_R = 2y_M$ ❶ $= 2a(p+q)$ $\quad = 2.\frac{a}{2}(p^2 + q^2)$ $= a(p^2 + q^2)$ So *R* is $(2a(p+q), a(p^2+q^2))$	Correct use of the midpoint formula.	Many lengthy and convoluted methods used for this one mark – usually without success.
iii. $x_R = 2a(p+q)$ and $y_R = a(p^2+q^2)$ Hence $(p+q) = \frac{x_R}{2a}$, and then $y_R = a(p^2 + q^2 + 2pq - 2pq)$ $= a((p+q)^2 - 2pq)$ $= a\left(\left(\frac{x_R}{2a}\right)^2 - 2.(-4)\right)$ ❶ $= a\left(\frac{x_R^2}{4a^2} + 8\right)$ $= \frac{x_R^2}{4a} + 8a$ So $x^2 = 4a(y - 8a)$ ❶	Both substitutions correct.	Many expanded the second line to $a(p+q)^2 - 2pq$, instead of $a(p+q)^2 - 2apq$ (failing to multiply the last term by *a*), thus losing a mark.

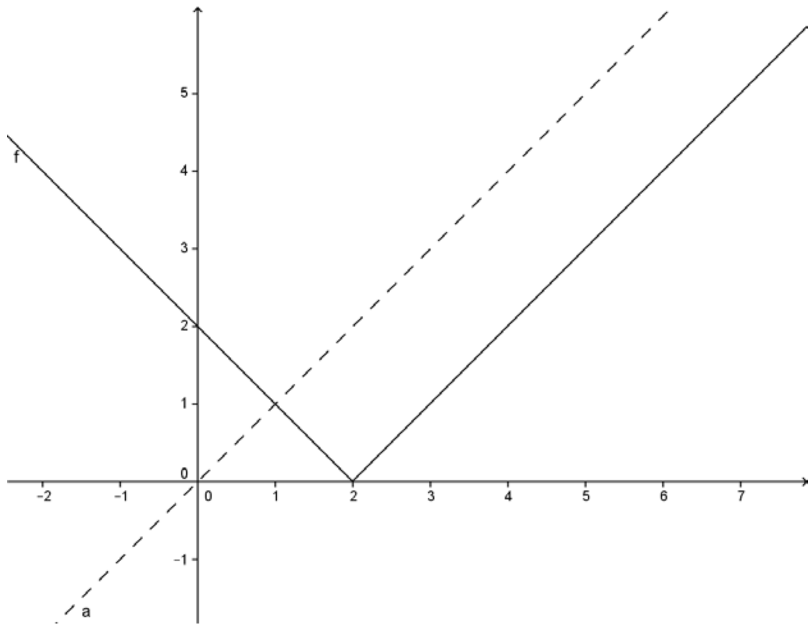
Question 12:

	Marking	Comments
a) $\alpha+\beta+\gamma=\frac{2}{3};\ \alpha\beta\gamma=\frac{4}{3}$ $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$ ❶ $=\frac{2}{3}\times\frac{3}{4}$ $=\frac{1}{2}$ ❶		Keep it simple. If you can't memorise $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}=\frac{\alpha+\beta+}{\alpha\beta\gamma}$ ..then find the common denominator.
b) $\lim_{h\to0}\left(\frac{\cos2h-1}{h}\right)\times\left(\frac{\cos2h+1}{\cos2h+1}\right)$ $=\lim_{h\to0}\left(\frac{\cos^2 2h-1}{h(\cos2h+1)}\right)$ $=\lim_{h\to0}\left(\frac{-\sin^2 2h}{h(\cos2h+1)}\right)$ $=\lim_{h\to0}\left(\frac{\sin2h}{2h}\times\frac{-2\sin2h}{(\cos2h+1)}\right)$ ❶ $=\lim_{h\to0}\left(\frac{\sin2h}{2h}\right)\times\lim_{h\to0}\left(\frac{-2\sin2h}{\cos2h+1}\right)$ $=1\times0$ $=0$ ❶	Alternative methods possible here	
Or $\lim_{h\to0}\left(\frac{\cos2h-1}{h}\right)=\lim_{h\to0}\left(\frac{1-2\sin^2 h-1}{h}\right)$ ❶ $=\lim_{h\to0}\left(\frac{2\sin^2 h}{h}\right)$ $=2\lim_{h\to0}\left(\frac{\sin h}{h}\right).\lim_{h\to0}(\sin h)$ $=2.1.0$ $=0$ ❶		$\frac{\sin^2 h}{h}\neq\frac{\sin h}{h}\frac{\sin h}{h}$ Remember $\lim_{h\to0}h=0\neq h$
Or $2\lim_{h\to0}\left(\frac{\sin h}{h}\right).\lim_{h\to0}(\sin h)$ $=2\lim_{h\to0}\left(\frac{\sin h}{h}\right).\lim_{h\to0}\left(\frac{\sin h}{h}\right)\lim_{h\to0}h$ $=2.1.1.0=0$ $\neq h$		

	Marking	Comments

c)

i.



❶ sketch of $y = |2 - x|$ correct.

ii. Comparing to $y = x$ (dashed line above), the graphs cross at $(1,1)$ ❶, hence $|2 - x| < x$ when $x > 1$. ❶

Marking: This should be explicit.

Comments:

Label the axes x & y.

Use arrows to indicate the positive direction and arrows on your rays/lines to indicate that they're not line segments.

Label intercepts with just the intercept value, not the full coordinates of the intercepts.

If solving algebraically, you must test your solutions. Solving algebraically often generates solutions which aren't valid, so they must be tested and eliminated.

Marking | Comments

d) Step 1: show true for $n=1$

$$LHS = 2 \quad RHS = \frac{n(3n+1)}{2}$$
$$= \frac{1(3+1)}{2}$$
$$= 2$$
$$= LHS$$

Hence true for $n=1$.

Step 2: Assume true for $n=k$:

i.e. assume $2+5+8+...+(3k-1)=\dfrac{k(3k+1)}{2}$

Then show true for $n=k+1$

i.e. show

$$2+5+8+...+(3k-1)+(3(k+1)-1)=\frac{(k+1)(3(k+1)+1)}{2}$$

❶ for setting up correctly.

Comments: Do not expand and simplify at this stage.

Step 3:

$$LHS = 2+5+8+...+(3k-1)+(3(k+1)-1)$$
$$= \frac{k(3k+1)}{2}+(3(k+1)-1)$$

(using the assumption ❶)

Marking: This should be explicit.

$$= \frac{k(3k+1)}{2}+3k+2$$
$$= \frac{k(3k+1)+6k+4}{2}$$
$$= \frac{3k^2+k+6k+4}{2}$$
$$= \frac{3k^2+7k+4}{2}$$
$$= \frac{3k^2+3k+4k+4}{2}$$
$$= \frac{3k(3k+1)+4(k+1)}{2}$$
$$= \frac{(k+1)(3k+4)}{2}$$

Comments: Most people stopped here. One line early.

$$= \frac{(k+1)(3(k+1)+1)}{2}$$ as reqd. ❶ (correct algebra to result)

Comments: Show the RHS result here.

Hence, by Mathematical Induction,

$$2+5+8+...+(3n-1)=\frac{n(3n+1)}{2}$$

e)

	Marking	Comments
i. $\frac{dT}{dt} = -k\left(T - S\right)$ $\frac{dT}{\left(T - S\right)} = -k\,dt$ Integrating: $\ln\left(T - S\right) = -kt + c$ $T - S = e^{-kt+c}$ $= Ae^{-kt}$ $T = S + Ae^{-kt}$ $T = 22 + Ae^{-kt}$ (as $S = 22$). When $t = 0, T = 55$, hence $55 = 22 + A.1$ $A = 33$ ❶ When $t = 10, T = 41$, hence $41 = 22 + 33e^{-10k}$ $\frac{19}{33} = e^{-10k}$ $-10k = \ln\left(\frac{19}{33}\right)$ $k = \frac{-1}{10}\ln\left(\frac{19}{33}\right)$ $= \frac{1}{10}\ln\left(\frac{33}{19}\right)$ ❶ Thus $T = 22 + 33e^{\frac{-t}{10}\ln\left(\frac{33}{19}\right)}$ as reqd.	Note: S is given – students just need to interpret this correctly.	Do not try to show that the solution fits the model by differentiating it's RHS. Many students stopped at $k = \frac{-1}{10}\ln\left(\frac{19}{33}\right)$ and fudged. Make k look positive (distribute the negative into the log) if you are working with a model which looks like this: $P = B + Ae^{-kt}$
ii. $t = 25$: $T = 22 + 33e^{\frac{-25}{10}\ln\left(\frac{33}{19}\right)}$ $= 30.3^{\circ}C$ ❶		

	Marking	Comments
iii. $T = 25$:		

$$25 = 22 + 33e^{\frac{-t}{10}\ln\left(\frac{33}{19}\right)}$$

$$\frac{3}{33} = e^{\frac{-t}{10}\ln\left(\frac{33}{19}\right)}$$

$$\ln\left(\frac{1}{11}\right) = \frac{-t}{10}\ln\left(\frac{33}{19}\right)$$ ❶

$$\frac{10\ln\left(\frac{1}{11}\right)}{\ln\left(\frac{33}{19}\right)} = -t$$

$$t = \frac{10\ln(11)}{\ln\left(\frac{33}{19}\right)}$$

$$= 43.43473528$$

$$\approx 43.4\,\text{min}$$ ❶

Question 13:

	Marking	Comments
a) $\frac{d}{dx}\left(e^{-x}\cos^{-1}x\right)$ $= -e^{-x}\cos^{-1}x - \frac{e^{-x}}{\sqrt{1-x^2}}$ ❶ ❶		Well done. **Note:** $\cos^{-1}x \neq \frac{1}{\cos x}$
b) With $u = x-2$ and $x=5$ $x=4$ $du = dx$ $u=3$ $u=2$ Then $I = \int_4^5 \frac{x(x-4)}{(x-2)}dx$ $= \int_2^3 \frac{(u+2)(u-2)}{u}du$ $= \int_2^3 \frac{u^2-4}{u}du$ $= \int_2^3 u - \frac{4}{u}du$ ❶ $= \left[\frac{u^2}{2} - 4\ln(u)\right]_2^3$ ❶ $= \frac{9}{2} - 4\ln 3 - \left(\frac{4}{2} - 4\ln 2\right)$ $= \frac{5}{2} - 4\ln\left(\frac{3}{2}\right)$ ❶		Generally well done. $\frac{5}{2} + 4\ln\frac{2}{3}$ also correct. Evaluate means 'give exact answer' i.e. don't approximate.

	Marking	Comments

c) $P(x) = x^3 + 2x^2 - 15x - 36$

i. Testing:

$P(1) = -48 \quad P(-1) = -20 \quad P(2) = -50 \quad P(-2) = 6$

$P(3) = -36 \quad P(-3) = 0$

Hence $x+3$ is a factor of $P(x)$. ❶

Marking: Variants possible depending on root chosen.

$$\begin{array}{r} x^2 - 1x - 12 \\ x+3 \overline{\big) x^3 + 2x^2 - 15x - 36} \\ x^3 + 3x^2 \\ -x^2 - 15x \\ -x^2 - 3x \\ -12x - 36 \\ -12x - 36 \\ \hline \end{array}$$ ❶

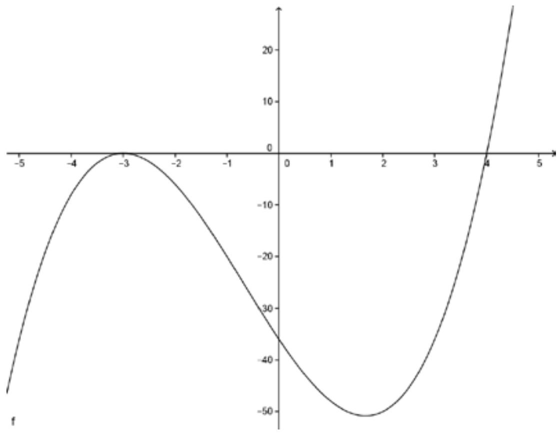
Comments: Should show this working but was not penalised if it wasn't. Generally very well done.

Now $x^2 - 1x - 12$

$= (x-4)(x+3)$

Hence $P(x) = (x-4)(x+3)^2$ ❶

ii. Thus for $x^3 + 2x^2 - 15x - 36 < 0$



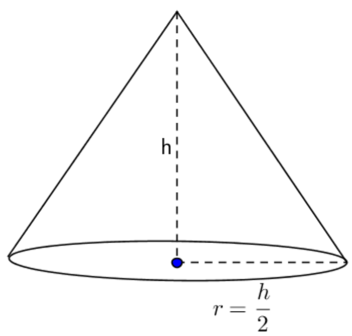
❶ (graph or equivalent justification)

Hence $x < -3 \, and -3 < x < 4$ ❶

Comments: A graph or testing regions needed to be shown to gain marks.

$x < 4, x \neq 4$ also acceptable answer.

d) Diagram:



	Marking	Comments
i. $V=\frac{1}{3}\pi r^2 h$		
$=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ ❶		Needed to show this line as it was a *'show'* question.
$=\frac{\pi}{12}h^3$		
$\therefore \frac{dV}{dh}=\frac{\pi}{4}h^2$		
Then		
$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$		
$20=\frac{\pi}{4}h^2\times\frac{dh}{dt}$		Well done.
$\frac{dh}{dt}=\frac{80}{\pi h^2}$ ❶		
ii. $\frac{dh}{dt}=\frac{80}{\pi h^2}$		
$=\frac{80}{\pi .8^2}$		
$=\frac{5}{4\pi}$ ❶		Well done.
iii. $\frac{dh}{dt}=\frac{80}{\pi h^2}$		
$\pi h^2\,dh=80\,dt$		Other valid methods accepted.
Integrating:		
$\frac{\pi}{3}h^3=80t+c$		
$t=0, h=0$, hence $c=0$.		One should show the evaluation of the integration constant, *c* but not penalised.
$\therefore \frac{\pi}{3}h^3=80t$		
$t=\frac{\pi h^3}{240}$ ❶		
When $h=8$:		A common error was to start with $\frac{dh}{dt}=\frac{5}{4\pi}$ which is a specific case rather than the general case. This led to getting an incorrect answer of 20.1 secs.
$t=\frac{\pi .8^3}{240}$		
$=\frac{32\pi}{15}$		
$=6.7\,\text{sec}$. ❶		

Question 14:

a) From $\ddot{x} = 4 - 4x$

i. $\ddot{x} = -4(x-1)$

$= -2^2(x-1)$

which is of the form $\ddot{x} = -n^2 x$ ❶, hence the motion is SHM with $n = 2$ and centre of motion at $x = 1$. ❶

ii. $\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 4 - 4x$

$\frac{1}{2}v^2 = 4x - 2x^2 + c$

$v^2 = 8x - 4x^2 + c$

When $x = 3$, $v^2 = 20$:

$20 = 8.3 - 4.3^2 + c$

$20 = 24 - 36 + c$

$c = 32$ ❶ hence

$v^2 = 32 + 8x - 4x^2$, as reqd.

iii. As motion is SHM, $v = 0$ when x is a min/max. Thus

$0 = 32 + 8x - 4x^2$

$= -4(x^2 - 2x - 8)$

$= -4(x-4)(x+2)$

$\therefore -2 \le x \le 4$ ❶

iv. The particle starts at $x = 3$, moving to the left and reaching a maximum velocity of $v = 6$ when $x = 1$, then decelerating to a stop at $x = -2$ ❶

b)

i. $\dfrac{d\left(x\sin^{-1}x\right)}{dx} = \sin^{-1}x + \dfrac{x}{\sqrt{1-x^2}}$ ❶

$\sin^{-1}x = \dfrac{d\left(x\sin^{-1}x\right)}{dx} - \dfrac{x}{\sqrt{1-x^2}}$

Integrating both sides:

$\int \sin^{-1}x\,dx = \int \dfrac{d\left(x\sin^{-1}x\right)}{dx}dx - \int \dfrac{x}{\sqrt{1-x^2}}dx$

$= x\sin^{-1}x - \int \dfrac{x}{\sqrt{1-x^2}}dx$ ❶

Marking

Solution/ description must include the effects crossing $x = 1$.

Comments

i. Generally answered well. Students need to explicitly state it's SHM because it's in the form $\ddot{x} = -n^2x$
A mark was deducted if both n and the C.O.M were not stated explicitly

ii. Generally answered well

iii. Answered poorly
Many students stated only values of x at the endpoints instead of a range of values

iv. Answered poorly. Position of particle at key points need to be stated along with velocity and acceleration.
Most gave insufficient answers

i. Not answered very well.
Students need to state the eqn. for evaluation. Don't include sign of integration in front of $x\sin^{-1}x$.

	Marking	Comments
ii. $\therefore \int_0^1 \sin^{-1} x \, dx = \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ $= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}} dx$ $= 1.\sin^{-1}(1) - 0.\sin^{-1}(0) + \frac{1}{2}.\frac{2}{1}\left[\sqrt{1-x^2}\right]_0^1$ ❶ $= \frac{\pi}{2} - 0 + (0-1)$ $= \frac{\pi}{2} - 1$ ❶		ii. Many careless errors. Some students were unable to correctly integrate the term $\frac{x}{\sqrt{1-x^2}} dx$
c) i. $x = Vt\cos\beta$ ❶ $y = h + Vt\sin\beta - \frac{1}{2}gt^2$ ❶		i. Answered well
ii. At time T: For object projected from origin O: $x_O = UT\cos\alpha \quad y_O = UT\sin\alpha - \frac{1}{2}gT^2$ For object projected from A: $x_A = VT\cos\beta \quad y_A = h + VT\sin\beta - \frac{1}{2}gT^2$ Also at this time T: $x_O = x_A \; and \; y_O = y_A$, thus (noting no V in the final required solution) $UT\cos\alpha = VT\cos\beta$ $V = \frac{UT\cos\alpha}{T\cos\beta}$ $= \frac{U\cos\alpha}{\cos\beta}$ {1} ❶ $UT\sin\alpha - \frac{1}{2}gT^2 = h + VT\sin\beta - \frac{1}{2}gT^2$ $UT\sin\alpha = h + VT\sin\beta$ $h = UT\sin\alpha - VT\sin\beta$ {2} ❶ {1}into {2}:		ii. Finding the Cartesian equation by eliminating T was not necessary since it appears in the eqn to be proven. h was added to the horizontal (x) component instead of the vertical. If students didn't explicitly show the expanded form of $\sin(\alpha - \beta)$ a mark was deducted

	Marking	Comments
$h = UT\sin\alpha - \left(\dfrac{U\cos\alpha}{\cos\beta}\right)T\sin\beta$ ❶		
$= T\left(U\sin\alpha - \dfrac{U\cos\alpha\sin\beta}{\cos\beta}\right)$		
$= TU\left(\dfrac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\cos\beta}\right)$		
$h\cos\beta = TU\left(\sin\left(\alpha-\beta\right)\right)$ ❶		
$T = \dfrac{h\cos\beta}{U\left(\sin\left(\alpha-\beta\right)\right)}$ as reqd.		

Question 11 (d) – using the transformation:

$$1+\cos 2x=\sin 2x$$
$$1=\sin 2x-\cos 2x$$

(i) $R\sin 2x\cos\theta-R\sin\theta\cos 2x=R\sin\left(2x-\theta\right)$

$R^2=1^2+1^2$ and $\tan\theta=\left(\frac{1}{1}\right)$

$R=\sqrt{2}$

$$\theta=\tan^{-1}\left(1\right)$$
$$=\frac{\pi}{4}❶$$

$$\sqrt{2}\sin\left(2x-\frac{\pi}{4}\right)=1$$
$$\sin\left(2x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$
$$2x-\frac{\pi}{4}=n\pi+\left(-1\right)^n\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)❶$$
$$2x=n\pi+\frac{\pi}{4}+\left(-1\right)^n\frac{\pi}{4}$$
$$x=n\frac{\pi}{2}+\frac{\pi}{8}+\left(-1\right)^n\frac{\pi}{8}$$
$$=\left(4n+1+\left(-1\right)^n\right)\frac{\pi}{8}①$$

with $n=0,\pm1,\pm2,\ldots$ for the general solution. With n even: $n=2m$ gives:

$$x=\left(8m+1+1\right)\frac{\pi}{8}$$
$$=\left(4m+1\right)\frac{\pi}{4}$$
$$=\ldots\frac{-3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\ldots$$
$$=\frac{\pi}{4},\frac{5\pi}{4}\left(\text{for }0\le x\le 2\pi\right)❶$$

For n odd, $n=2m+1$ gives:

$$x=\left(4\left(2m+1\right)+1-1\right)\frac{\pi}{8}$$
$$=\left(2m+1\right)\frac{\pi}{2}$$
$$=\ldots\frac{-3\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots$$
$$=\frac{\pi}{2},\frac{3\pi}{2}\left(\text{for }0\le x\le 2\pi\right)❶$$

$$1+\cos 2x=\sin 2x$$
$$\cos 2x-\sin 2x=-1$$

(ii) $R\cos 2x\cos\theta+R\sin 2x\sin\theta=R\cos\left(2x+\theta\right)$

$R^2=1^2+1^2$ and $\tan\theta=\left(\frac{1}{1}\right)$

$R=\sqrt{2}$

$$\theta=\tan^{-1}\left(1\right)$$
$$=\frac{\pi}{4}❶$$

$$\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right)=-1$$
$$\cos\left(2x+\frac{\pi}{4}\right)=\frac{-1}{\sqrt{2}}$$
$$2x+\frac{\pi}{4}=2n\pi\pm\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)❶$$
$$2x=2n\pi-\frac{\pi}{4}\pm\frac{3\pi}{4}$$
$$x=n\pi-\frac{\pi}{8}\pm\frac{3\pi}{8}$$
$$=\left(8n-1\pm3\right)\frac{\pi}{8}②$$

with $n=0,\pm1,\pm2,\ldots$ for the general solution.

$$x=\ldots\frac{-4\pi}{8},\frac{2\pi}{8},\frac{4\pi}{8},\frac{10\pi}{8},\frac{12\pi}{8},\frac{18\pi}{8},\ldots$$
$$=\frac{\pi}{4},\frac{\pi}{2},\frac{5\pi}{4},\frac{3\pi}{2}\left(\text{for }0\le x\le 2\pi\right)❶❶$$

Please note:

- how much easier a transformation into *cos* is compared to *sin*
- ① and ② are sometimes referred to as ***generators*** for the fractional value of π in the general solutions (again, note the easier version that *cos* has over *sin*). Some students attempted to use the generators, only a couple were successful – you must pay attention to the required range. Successful students usually used something such as:

$$0\le x\le 2\pi$$
$$0\le 2x\le 4\pi$$
$$\frac{-\pi}{4}\le 2x-\frac{\pi}{4}\le 4\pi-\frac{\pi}{4}$$

to ensure their answers fell into the required range.